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NRL/MR/5540--10-9248

Comments on Optical Fiber Communication Channel Capacity Results of Song, Mahajan, Mahadevan, and Morris

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March 10, 2010

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
10-03-2010

2. REPORT TYPE
Memorandum Report

3. DATES COVERED *(From - To)*
1 June 2009 - 4 December 2009

4. TITLE AND SUBTITLE

Comments on Optical Fiber Communication Channel Capacity Results of Song, Mahajan, Mahadevan, and Morris

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
61153N

6. AUTHOR(S)

Ira S. Moskowitz and Daniel D. Kang

5d. PROJECT NUMBER

5e. TASK NUMBER
IT015-09-41

5f. WORK UNIT NUMBER
4288

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Research Laboratory, Code 5540
4555 Overlook Avenue, SW
Washington, DC 20375-5320

8. PERFORMING ORGANIZATION REPORT NUMBER

NRL/MR/5540--10-9248

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Naval Research
875 North Randolph Street
Arlington, VA 22203-1995

10. SPONSOR / MONITOR'S ACRONYM(S)

ONR

11. SPONSOR / MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

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15. SUBJECT TERMS

Information theory — Binary symmetric channel
Capacity — Asymmetric error

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON / 19b. TELEPHONE NUMBER *(include area code)*
Unclassified	Unclassified	Unclassified	UL	13	Ira S. Moskowitz / (202) 404-7930

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

COMMENTS ON OPTICAL FIBER COMMUNICATION CHANNEL CAPACITY RESULTS OF SONG, MAHAJAN, MAHADEVAN, AND MORRIS

IRA S. MOSKOWITZ AND DANIEL D. KANG

Abstract. We report on our numerical results concerning the capacity of certain binary symmetric channels with asymmetric erasures. Plots of our numerical capacity solutions are offered in contrast to some of the existing literature.

1. Introduction

We report on some numerical discrepancies in the plots from [9] that were noted by Daniel Kang when he interned at the Naval Research Laboratory as a SEAP student during the summer of 2009. In addition, the seeming impossibility of obtaining closed form solutions for the channel capacity in this paper has served as motivation for the capacity bounds illustrated in [7]. The results in this report were obtained by using the Blahut-Arimoto algorithm for numerically calculating the capacity [4].

2. Optical Fiber Communication

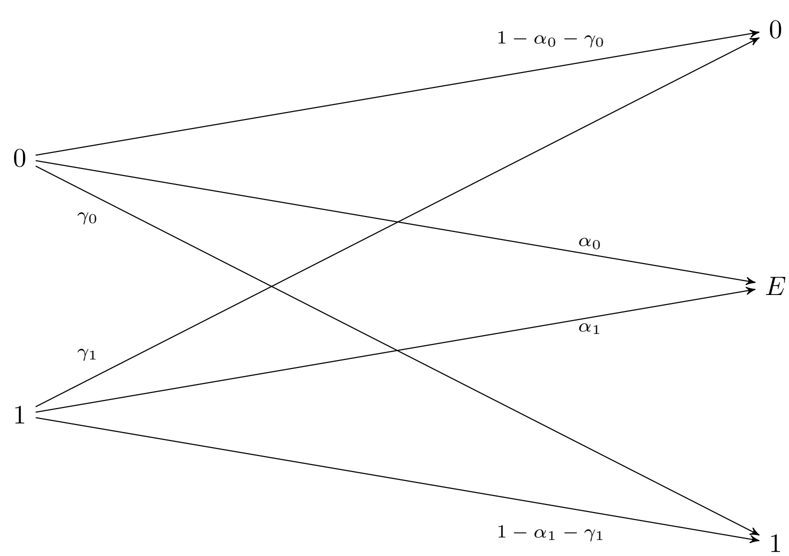

Figure 1: Channel transition diagram for BAC/AE channel

Manuscript approved December 17, 2009.

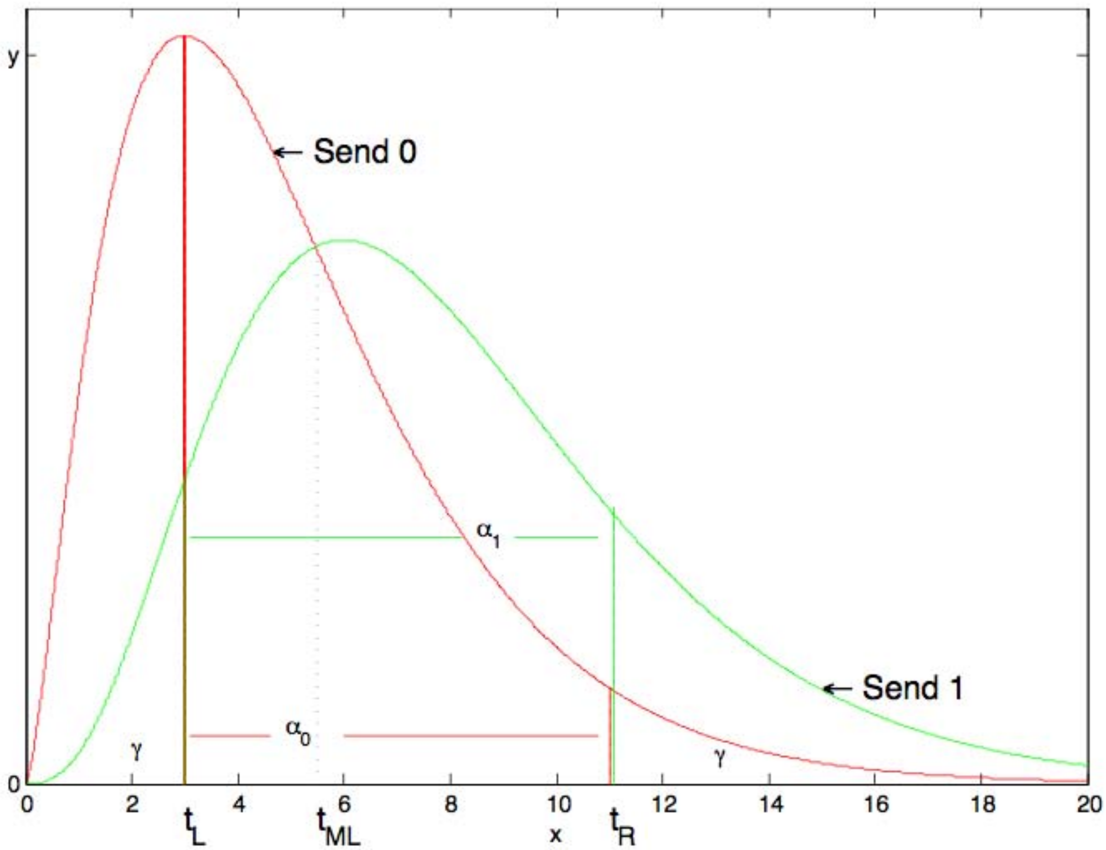

FIGURE 2: BSC/AE channel χ^2 probability density functions.

Most of this section is a description of what is given in [9]. Therefore, we will freely paraphrase the theoretical background as given in [9], with the understanding that this section borrows heavily, without quotation marks (for better readability), what is given in [9]. Additional information on optical fiber communication can also be found in [10]. (Note, though, sometimes we emphasize the reliance on [9] by explicitly calling out the reference.)

We wish to send an optical pulse. Receipt of the pulse is considered a *mark* and interpreted as the symbol 1. Non-receipt of the pulse is considered to be a *space* and is interpreted as the symbol 0. If this were all there was to it (that is, we transmit in a perfect world) we would have a discrete memoryless channel with binary input and binary output (a $(2, 2)$ channel for short). The physical process of sending light involves various mathematical models, physical limitations and inherent noise. Additionally, the mark and space are modeled by different probability distributions.

In an optical fiber channel, light is sent and a photodiode at the detecting end converts the light into a voltage. Since the photodetector acts ([9]) as a square-law detector, it is modeled as a χ^2, not a Gaussian, distribution. It is the value of the voltage that all decisions, hard and soft, are made against. In the model under consideration, it is possible for there to be a bit error in what is received; the transmitter may wish to send a 0, but a 1 is received, or it is desired to send a 1, but a 0 is received. There is also the possibility of symbol ambiguity where the receiver does not attempt a determination as to what was sent—this is called an erasure.

The most general model is the binary asymmetric channel with asymmetric erasures (BAC/AE); see *Figure 1. Figure 2* gives guidance as to how a symmetric error probability γ, and the erasure probabilities α_0, α_1, relate to the probability densities of sending a 0 or a 1. The simpler binary asymmetric channel (BAC) model is illustrated in [9]; this is a $(2,2)$ channel. In the BAC, there are no erasures, and the error terms are not equal and are usually taken as the integral of the distribution for sending a 0, between t_{ML} and ∞, and the integral of the distribution for sending a 1, between 0 and t_{ML}. The value t_{ML} is where the two distributions intersect (we will return to this later with a caveat). The distributions under study are single-humped, so t_{ML} is well defined.

Thus, our channel, instead of being a $(2,2)$ channel, is a $(2,3)$ channel. In [9] it is discussed that, through various assumptions and flexibility with respect to the soft decision thresholds, it suffices to study a binary *symmetric* channel with asymmetric erasures (BSC/AE). Note that the BAC/AE channel with soft decision making offers higher capacity than the similar BAC [9]. Starting with a BAC/AE channel one cannot make both the γ_i equal and the α_j equal. The BSC/AE channel model makes the γ_i equal, and then calculates α_0 and α_1 (we return to this later).

In a BSC/AE channel, the probability that a transmitted 0(1)is interpreted as a 1(0) is γ. The probability that a transmitted 0(1) cannot be decided by the receiver and is taken to be an erasure E is $\alpha_0(\alpha_1)$. They are determined by various voltage values (if just t_{ML}, the value where the distributions agree, were used as a cut-off for the errors with no erasures, we would have the binary erasure (BAC) $(2,2)$ channel).

Sending a 0, corresponds to a "space." In a perfect world, there would be no voltage received. However, the transmission process involves sums of zero-mean Gaussian random variables, plus there is also transmission noise. Due to this the process of sending a 0 is modeled as a central χ^2 distribution, where the degrees of freedom depend on the dimensionality of the transmitted signal space. This probability density is $p(x|0)$, with x being the received value conditioned on a 0 being sent.

To send a 1, which corresponds to a "mark," we transmit a narrow band Gaussian process that, when combined with signal noise, results in a non-central χ^2 distribution. This probability density is $p(x|1)$, with x being the received value conditioned on a 1 being sent.

There are two important physical constants for the optical fiber communication channel under review in this report. The constant M is the degrees of freedom for the underlying χ^2 model. The constant β is the signal to noise ratio, given in dB. We have $\beta = R_c E_b/N_0$, where R_c is the code rate, E_b is the energy per information bit (that is a mark), and N_0 is the power spectral density of the amplified spontaneous emission noise.

After normalizing for the various physical terms in the optical fiber communication, via the constants β and M, the distributions are given by their probability density functions as [9]

$$p(x|0) = \frac{x^{M-1}e^{-x}}{(M-1)!}, \text{ and} \tag{2.1}$$

$$p(x|1) = \left(\frac{x}{\beta}\right)^{\frac{M-1}{2}} e^{-x}e^{-\beta} I_{(M-1)}\left(2\sqrt{x\beta}\right), \tag{2.2}$$

where $I_{(M-1)}$ is the modified Bessel function of the first kind of order $M-1$.

Our goal, as in [9], is to calculate the capacity of the BSC/AE channel, given values for M and β, against the symmetric error probability γ. To do this we take a γ value and (numerically) determine values $t_L \le t_{ML}$ and $t_R \ge t_{ML}$ by

$$\gamma = \int_0^{t_L} \left(\frac{x}{\beta}\right)^{\frac{M-1}{2}} e^{-x}e^{-\beta} I_{(M-1)}\left(2\sqrt{x\beta}\right) dx = \int_{t_R}^{\infty} \frac{x^{M-1}e^{-x}}{(M-1)!} dx . \tag{2.3}$$

Then, once we have the t_L and t_R values, we can determine the erasure probabilities α_0 and α_1 by

$$\alpha_0 = \int_{t_L}^{t_R} \frac{x^{M-1}e^{-x}}{(M-1)!} dx \tag{2.4}$$

$$\alpha_1 = \int_{t_L}^{t_R} \left(\frac{x}{\beta}\right)^{\frac{M-1}{2}} e^{-x}e^{-\beta} I_{(M-1)}\left(2\sqrt{x\beta}\right) dx . \tag{2.5}$$

In [9], various values of M and β are discussed, but graphical results are given only for $M = 3$ and $\beta = 1, ..., 18$. Therefore, we restrict our analysis to the same in this report.

Note, for $M = 3$ the right hand side of *Equation (2.3)* simplifies via integration by parts. That is, since

$$\begin{aligned}
\int \frac{x^2 e^{-x}}{2} dx &= \frac{1}{2}\left\{-x^2e^{-x} + \int 2xe^{-x} dx\right\} \\
&= \frac{1}{2}\left\{-x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x} dx\right)\right\} \\
&= \frac{1}{2}\left\{-x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right)\right\} + \text{constant} .
\end{aligned}$$

We have that

$$\gamma = e^{-t_R}\left(\frac{(t_R)^2 + 2t_R + 2}{2}\right),$$

since as $x \to \infty$, $e^{-x} \to 0$ faster than a polynomial of x grows. However, this still does not give us a closed form for t_R, so we may either numerically integrate or use the above to obtain t_R. To obtain t_L, which additionally depends on β, *we* must numerically integrate. Once t_R and t_L are obtained, we integrate *Equations (2.4) & (2.5)* to obtain α_0 and α_1.

The channel matrix is

$$M = \begin{pmatrix} 1-\alpha_0-\gamma & \alpha_0 & \gamma \\ \gamma & \alpha_1 & 1-\alpha_1-\gamma \end{pmatrix}.$$

The channel input random variable X is defined so that the probability that a 0 is sent is x and the probability that a 1 is sent is $1-x$, and let the output random variable Y takes on the values 0, E, and 1. Keep in mind that $H(p_1, ..., p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i$ is the entropy function [8]. We have that

$$\begin{aligned} P(Y=0) &= P(Y=0|X=0)x + P(Y=0|X=1)(1-x) \\ &= (1-\alpha_0-\gamma)x + \gamma(1-x) \\ &= (1-\alpha_0-2\gamma)x + \gamma \ , \end{aligned}$$

$$\begin{aligned} P(Y=E) &= P(Y=E|X=0)x + P(Y=E|X=1)(1-x) \\ &= \alpha_0 x + \alpha_1(1-x) \\ &= (\alpha_0-\alpha_1)x + \alpha_1 \ , \end{aligned}$$

$$\begin{aligned} P(Y=1) &= P(Y=1|X=0)x + P(Y=1|X=1)(1-x) \\ &= \gamma x + (1-\alpha_1-\gamma)(1-x) \\ &= (-1+\alpha_1+2\gamma)x + 1 - \alpha_1 - \gamma \ , \text{ so} \end{aligned}$$

$$H(Y) = P(Y=0)\log_2 P(Y=0) + P(Y=E)\log_2 P(Y=E) + P(Y=1)\log_2 P(Y=1) \ .$$

$$\begin{aligned} H(Y|X) &= H(Y|X=0)x + H(Y|X=1)(1-x) \ , \text{ where} \\ H(Y|X=0) &= H(1-\alpha_0-\gamma, \alpha_0, \gamma) \ , \text{ and} \\ H(Y|X=1) &= H(\gamma, \alpha_1, 1-\alpha_1-\gamma) \ , \text{ and the mutual information is} \end{aligned}$$

$$\begin{aligned} I(X,Y) &= H(Y) - H(Y|X) \ . \text{ Hence} \\ I(X,Y) &= H\big((1-\alpha_0-2\gamma)x+\gamma, (\alpha_0-\alpha_1)x+\alpha_1, (-1+\alpha_1+2\gamma)x+1-\alpha_1-\gamma\big) \\ &\quad -xH\left(1-\alpha_0-\gamma, \alpha_0, \gamma\right) - (1-x)H\left(1-\alpha_1-\gamma, \alpha_1, \gamma\right) \ . \end{aligned}$$

The channel capacity [8] is

$$C = \max_x I(X,Y) \ . \tag{2.6}$$

Keep in mind that we are dealing with a $(2,3)$ channel. For a $(2,2)$ channel, the maximization of the mutual information is a simple calculus problem and the problem has been well studied (e.g. [2, 5]). However, for a $(2,3)$ channel there is no closed form solution in general for the capacity. Therefore, numerical techniques have been relied on for the capacity. The Blahut-Arimoto algorithm [4, pp. 364-367],[3, 1] was used to numerically determine the capacity.

We attempt to replicate the plots given in [9]. As noted before, our plots are somewhat different that those in [9].

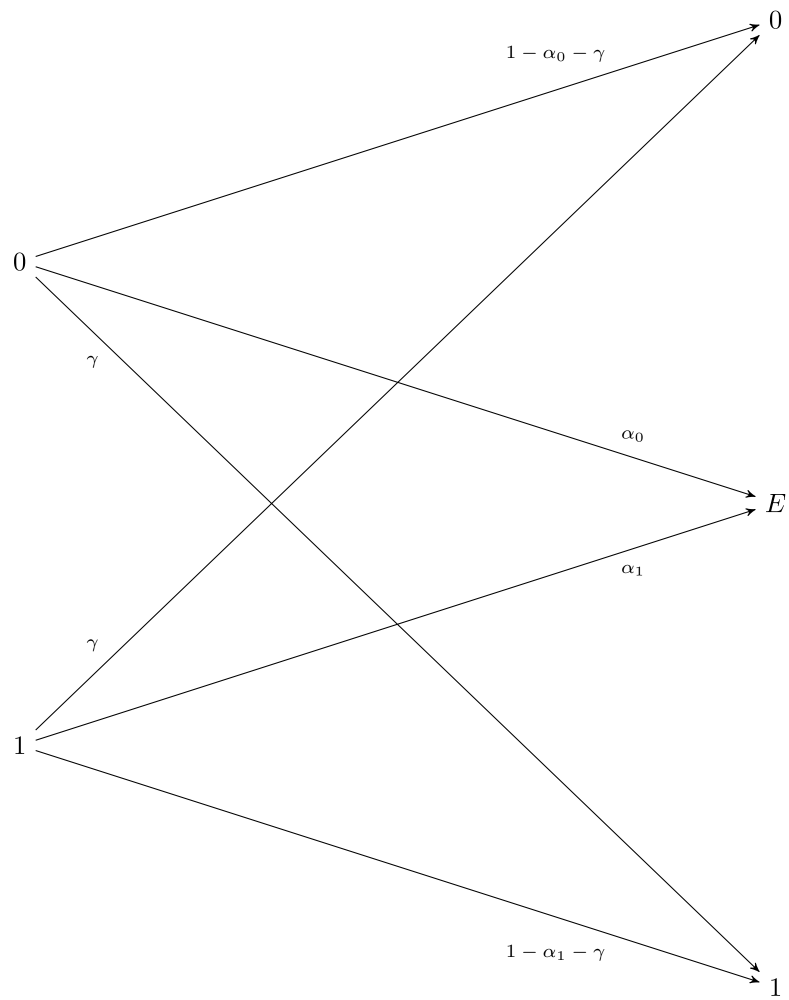

Figure 3: Channel transition diagram for BSC/AE channel

3. Plots

In this section, we present the graphical output of our plots. As noted, we offer these as corrections as to what was put forth in [9]. We note that in [9] the definition of the test condition t_{ML} was slightly adjusted in practice, even though suboptimal detection is achieved, from how it was theoretically presented. Ideally, as discussed above, t_{ML} is the non-trivial x value where $p(x|o) = p(x|1)$. In [9], for the situation under consideration in this report, t_{ML} is actually the x value where $\int_0^{t_{ML}} p(x|1)dx = \int_{t_{ML}}^{\infty} p(x|0)dx$. We ran our plots using both definitions. The only difference is that the second definition allows slightly larger γ values, there is insignificant difference in the plots. Therefore we stay with the later definition. We ask the the reader to keep in mind that the only use of t_{ML} in this report is as the test condition $t_L \leq t_{ML} \leq t_R$.

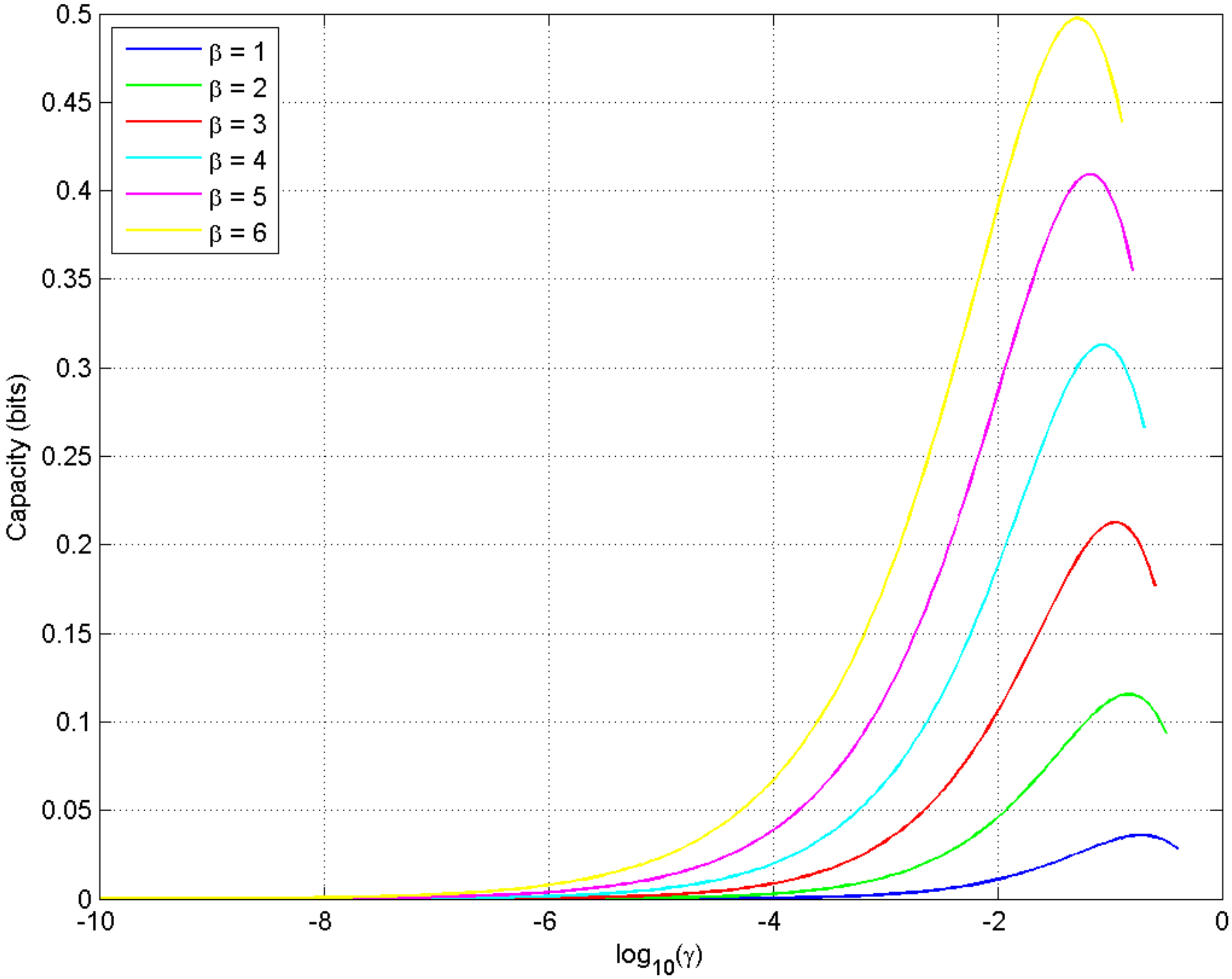

FIGURE 4: Capacity vs. γ for $\beta = 1, ..., 6$ and $M = 3$.

Figures 4, 5, & 6 show the plots of capacity, for M fixed at 3 and β ranging from 1 to 18. The x-axis is the value of the error probability γ using logarithmic scaling. The plots are of capacity vs. γ for $M = 3$, and $\beta = 1, ..., 18$. Even though the channel matrix is given in terms of γ, α_0, and α_1, the physical interest is also in seeing how capacity depends on β, see [9].

When comparing our *Figure 4* to [9, Fig. 15] we see that the overall shape is the same. The difference is that our maximums are larger and our plots decrease past the maximum.

When comparing our *Figure 5* to [9, Fig. 16] we see that the overall shape is the same. As for the lower β values the difference is that our maximums are larger and our plots decrease past the maximum. However, when comparing our *Figure 6* to [9, Fig. 17] we see a marked difference. As $\beta \to 18$ our plots have the same shape as for lower value β. However, in [9, Fig. 17] the plots approach a constant value of 1, as β grows. We attribute the differences to numerical errors in [9]. We offer theoretical reasons why our plots are reliable below.

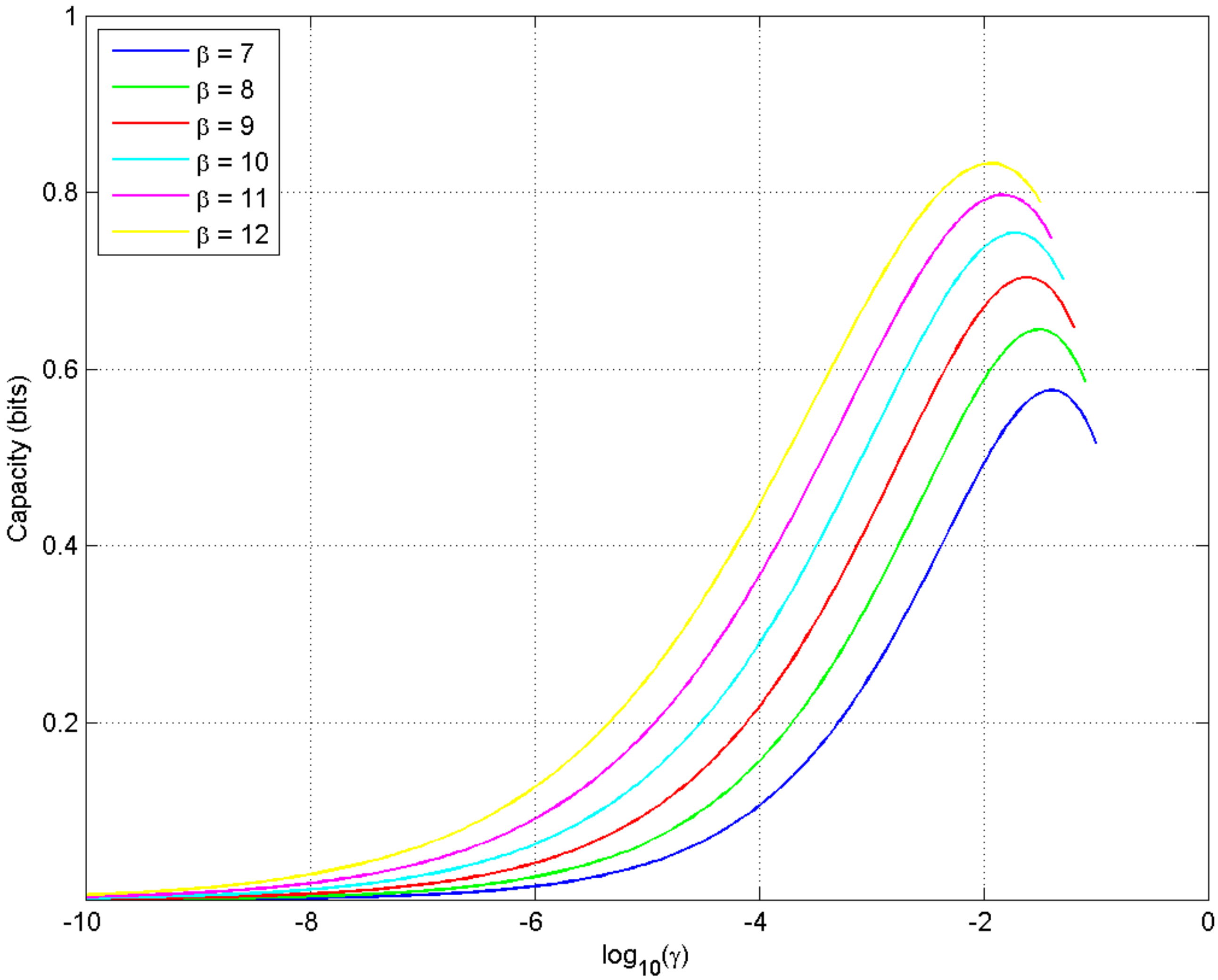

FIGURE 5: Capacity vs. γ for $\beta = 7, ..., 12$ and $M = 3$

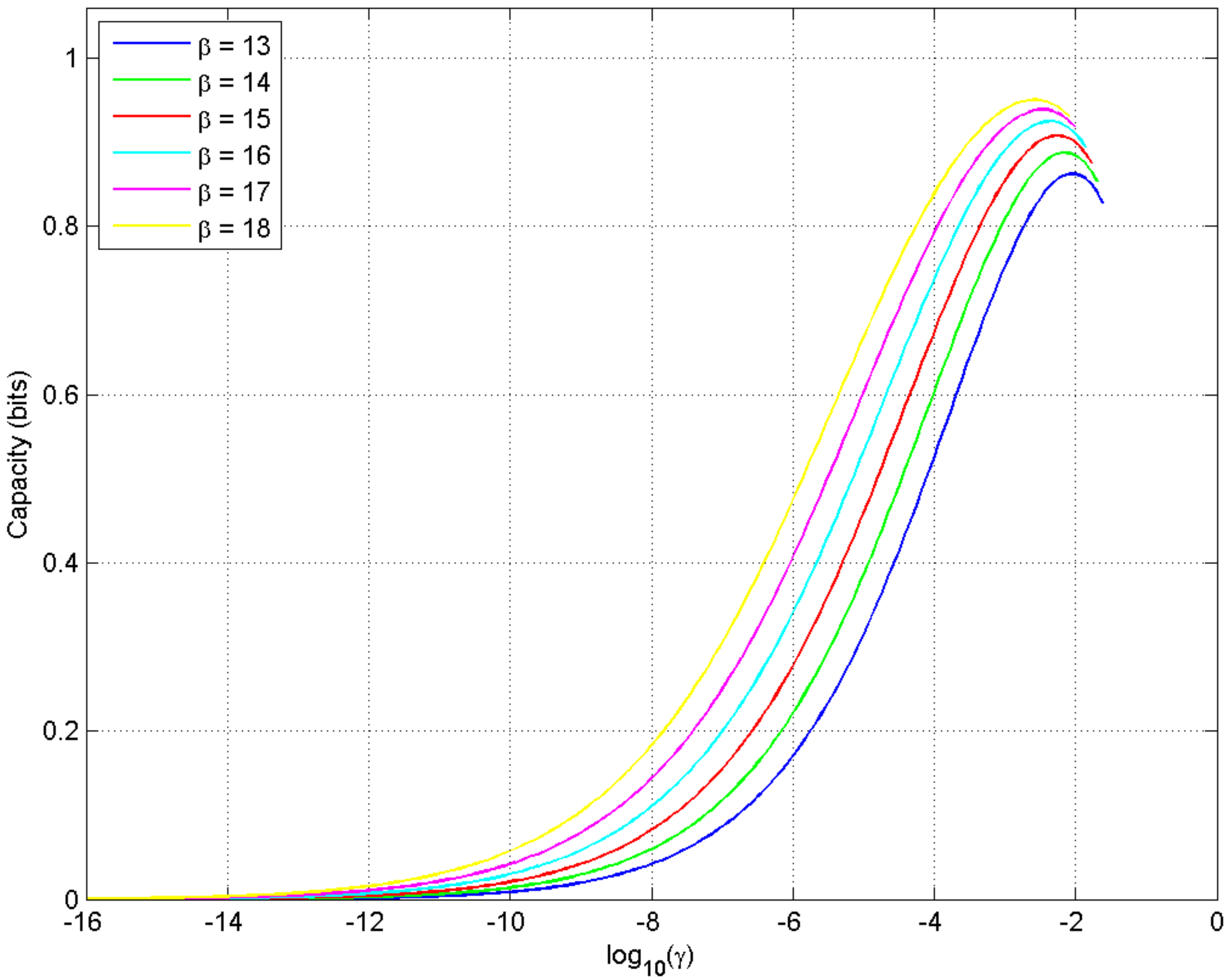

FIGURE 6: Capacity vs. γ for $\beta = 13, ..., 18$ and $M = 3$

4. Capacity Behavior and Future Research

It is interesting to see that capacity behaves in a slightly *humped* manner. On a logarithmic scale, the capacities start off near 0 for very small γ. The larger β, the smaller γ is until the capacity is essentially 0. As $\gamma \to 0^+$, both α_0 and α_1 approach 1, so

$$M \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

hence $C \to 0$ as the plots show.

The capacity increases as γ grows, and then slightly decreases as γ grows to its upper limit. Specifically, capacity decreases and falls off slightly as $\gamma \to \approx 10^{-.43^{-}}$, for $\beta = 1$, and upper limiting to $\approx 10^{-2.1}$, for $\beta = 18$. The upper limits of γ are determined by the requirement $t_L \leq t_{ML} \leq t_R$.

We wish to investigate why the plots seem to peak just shy of their upper limits, and the physical implications of this. There is no closed form solution for the capacity of a generic $(2,3)$ channel, but we would still like to glean some knowledge of the capacity behavior via direct examination of the channel matrix. For a $(2,2)$ channel, this approach closely tracks the behavior of the capacity. Specifically, in [6], it is shown that the capacity $C_{2,2}$ for a $(2,2)$ channel with channel matrix

$$M = \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}$$

obeys

$$\frac{(a-b)^2}{2\ln(2)} \leq C_{2,2} \leq |a-b| \tag{4.1}$$

This result has been generalized, and for a $(2,3)$ channel with channel matrix

$$M = \begin{pmatrix} a_1 & a_2 & \bar{a} \\ b_1 & b_2 & \bar{b} \end{pmatrix}$$

it has been shown in [7] that the capacity $C_{2,3}$ is bounded as:

$$\frac{1}{8\ln(2)}(|a_1-b_1|+|a_2-b_2|+|\bar{a}-\bar{b}|)^2 \leq C_{2,3} \leq \max(a_1+a_2, b_1+b_2)-\min(a_1,b_1)-\min(a_2,b_2). \tag{4.2}$$

We hope to use *Equation (4.2)*, along with more information on how α_0 and α_1 depend on γ to attempt to derive rule-of-thumb capacity results for the finer optical communication channel under study in this report.

5. Acknowledgements

We are extremely grateful to Joel M. Morris for his helpful information and comments. We are also grateful to Gerard Allwein, Ben Hosp and Ruth Irene for their assistance.

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